Lecture 8: Implicit Differentiation

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circle

Implicit differentiation

What if a function is defined as $x^2 + y^2 = 25$?

In some cases: express $y = \pm\sqrt{x^2 - 25}$ ← 2 functions

We want to differentiate y:

$y'(x) = \pm\frac{1}{2\sqrt{x^2-25}} * 2x$

or

$x^2 + y^2 = 25 \qquad y = y(x)$ a function in x

We can differentiate both sides with respect to x.

$\frac{d}{dx} = \left(x^2 + y^2(x)\right) = \frac{d}{dx}(25)$

$\frac{d}{dx}x^2 + \frac{d}{dx}y^2(x) = \frac{d}{dx}(25)$

$2x + 2 * y(x) * \underbrace{y'(x)}_{\text{inner derivative}} = 0$

$2x + 2y * y' = 0$ --> now solve for y'

$2y * y' = -2x$

$y * y' = -x$

$y' = -\frac{x}{y}$

Compute the tangent line at (3,4)

slope: $y' = -\frac{3}{4}$

equation for tangent line: $t(x)$

$t - 4 = -\frac{3}{4}(x - 3)$

$t = -\frac{3}{4}x - \frac{3}{4} * (-3) + 4$

$t = -\frac{3}{4}x + \frac{25}{4}$

Example

$x^3 + y^3 = 6xy$ Want to use implicit differentiation

Differentiate both sides:

$3x^2 + 3y^2 * y' = 6y + 6xy'$

solve for y' by putting all terms with y' on the left and the rest on the right side

$3y^2y' - 6xy' = 6y - 3x^2$

$y^2y' - 2xy' = 2y - x^2$

$y'(y^2 - 2x) = 2y - x^2$

$y' = \frac{2y-x^2}{y^2-2x}$

Tangent line at (3,3)

$y' = \frac{2(3)-3^2}{2^2-2(2)}$

$= \frac{6-9}{9-6} = -\frac{3}{3} = -1$

$t - 3 = -1(x - 3)$

$t = -x + 6$

Ex

$\sin(x + y) = 2x - 2y$

$\cos(x + y) * (1 + y') = 2 - 2y'$

$\cos(x + y) + y'\cos(x + y) = 2 - 2y'$

$y'(\cos(x + y) + 2) = 2 - \cos(x + y)$

$y' = \frac{2-\cos(x+y)}{\cos(x+y)+2}$

Tangent line at (pi, pi)

$$y'(pi) = \frac{2-\cos(pi+y(pi))}{\cos(x+y(pi))+2}$$

$$= \frac{2-\cos(2pi)}{\cos(2pi)+2} = \frac{2-1}{1+3} = \frac{1}{3}$$

slope $= 1/3$ at (π, π)

Derivatives of inverse trigonometric functions

$y(x) = \sin^{-1}(x)$ $\neq \frac{1}{\sin(x)}$

means: $\sin(y) = x$ use implicit differentiation

$\cos(y) * y' = 1$

$y' = \frac{1}{\cos(y)}$

if $\frac{-\pi}{2} < y < \frac{\pi}{2}$ $\Rightarrow \cos(y) > 0$

NB

recall: $\sin^2(y) + \cos^2(y) = 1 \Rightarrow \cos(y) = \sqrt{1-\sin^2(x)}$

$\sin^2(y) = x^2 \Rightarrow (\sin^{-1}(y))' = \frac{1}{\sqrt{1-x^2}}$

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

Example:

$\tan^{-1}(x^2y) = x + xy^2$, find y'

$$\frac{1}{1+(x^2y)^2} * (2xy + x^2y') = 1 + \underbrace{y^2 + x * 2y * y'}_{\text{product rule on } xy^2}$$

product rule on x^2y

$$\frac{2xy}{1+(x^2y)^2} + \frac{x^2y'}{1+(x^2y)^2} = 1 + y^2 + x2yy'$$

$$y'\left(\frac{x^2}{1-(x^2y)^2} - 2xy\right) = 1 + y^2 - \frac{2xy}{1+(x^2y)^2}$$

$$y' = \frac{1+y^2 - \frac{2xy}{1+(x^2y)^2}}{\frac{x^2}{1-(x^2y)^2} - 2xy}$$

simplify for practice

Derivatives of logarithmic functions

If $y = \ln(x)$ <=> $x = e^y$ ← implicit diff

$x = e^y$

$1 = e^y y'$ ← inner derivative

$$y' = \frac{1}{e^y} = \frac{1}{x}$$

$$(\ln(x))' = \frac{1}{x}$$ "derivative of natural logarithm is 1/x"

Examples:

- $y(x) = \ln(x^3+1)$

  $y'(x) = \frac{1}{x^3+1} * 3x^2$ ← inner derivative

$$= \frac{3x^2}{x^3 + 1}$$

- $f(x) = \sqrt{\ln(x)}$ (chain rule)

  $$f'(x) = \frac{1}{2\sqrt{\ln(x)}} * (\ln(x))' * (x)'$$
  $$= \frac{1}{2\sqrt{\ln(x)}} * \frac{1}{x} * 1$$
  $$= \frac{1}{2x\sqrt{\ln(x)}}$$

- $f(t) = (\ln(t))^2 * \sin(t)$ → product rule on $(\ln(t))^2$ and $\sin(t)$

  $$= 2\ln(t) * (\ln(t))' * \sin(t)$$
  $$+ (\ln(t))^2 * \cos(t)$$
  $$= 2\ln(t) * \frac{1}{t} * \sin(t) + (\ln(t))^2 * \cos(t)$$

- $f(x) = \dfrac{\ln(x) + \sin(x)}{3x^2 + 1}$

  $$f'(x) = \frac{\left(\frac{1}{x} + \cos(x)\right) * (3x^2 + 1) - (\ln(x) + \sin(x)) * (3 * 2x)}{(3x^2 + 1)^2}$$
  $$= \frac{3x + \frac{1}{x} + 3x^2\cos(x) + \cos(x) - 6x * \ln(x) - 6x * \sin(x)}{(3x^2 + 1)^2}$$

- $f(t) = \sqrt{\dfrac{3}{t} + 2t}$

  $$f'(t) = \frac{1}{2 * \sqrt{\frac{3}{t} + 2t}} * \left(3 * \left(-\frac{1}{t^2}\right) + 2\right)$$
  $$= \frac{-\frac{3}{t^2} + 2}{2\sqrt{\frac{3}{t} + 2t}} = \frac{\frac{-3 + 2t^2}{t}}{2\sqrt{\frac{3 + 2t^2}{t}}}$$

  after simplifying:

  $$= \frac{-3 + 2t^2}{t^{\frac{3}{2}} * 2 * \sqrt{3 + 2t^2}}$$

- $f(x) = e^{\cos(5x^2)}$ chain rule twice!

  $$f'(x) = e^{\cos(5x^2)} * (-\sin(5x^2) * 5 * 2x)$$